Exam Special Relativity
December 16, 2013
Start: 15:00h End: 17:00h

INSTRUCTIONS: This is a closed-book and closed-notes exam. The exam duration is 2 hours. You are allowed to use a numerical (non programmable) calculator. There is a total of 9 points that you can collect. Problems are designed, as much as it is possible, so that you can answer a given part of the problem without necessarily answer other parts. Work by default with SR units.

NOTE: If you are not asked to **Show your work**, then an answer is sufficient. However, you might always earn more points by answering more extensively (but you can also loose points by adding wrong explanations). If you are asked to **Show your work**, then you should explain your reasoning and the mathematical steps of your derivation in full. Use the official exam paper for *all* your work and ask for more if you need. Write clearly, and draw clearly the spacetime diagrams making use of the squared paper. **Solve the problems on separate sheets**, e.g. problem 1 on sheet 1 and problem 2 on sheet 2.

USEFUL FORMULAS AND CONSTANTS

$$3.00 \times 10^8\,\mathrm{m} = 1\mathrm{s} \qquad (1+x)^n \sim 1 + nx \quad x \ll 1$$

$$m^2 = E^2 - |\vec{P}|^2 = E^2 - P_x^2 - P_y^2 - P_z^2 \qquad P_x = \frac{m v_x}{\sqrt{1-v^2}} \qquad E = \frac{m}{\sqrt{1-v^2}}$$

$$\Delta t = \gamma \Delta t' + \gamma\beta\Delta x' \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\Delta x = \gamma \Delta x' + \gamma\beta\Delta t'$$

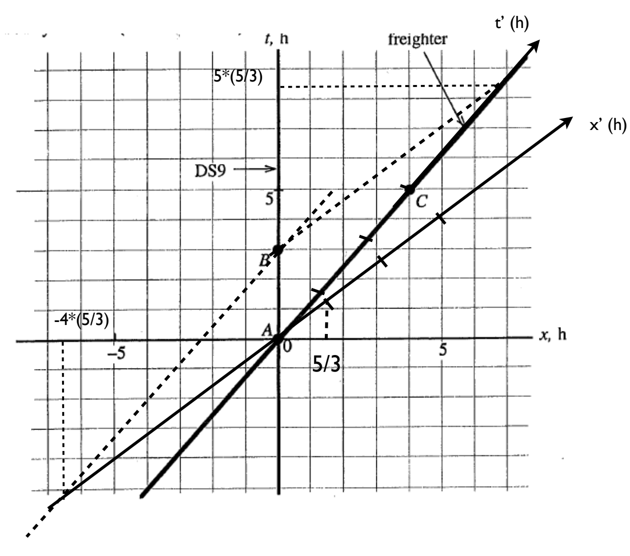
$$\Delta y = \Delta y'$$

$$\Delta z = \Delta z'$$

1. (5 points total) Consider the spacestation DS9 at rest in the solar system. The latter defines the Home Frame, and use DS9 as the origin of the Home Frame. A Ferengi freighter moves at a constant speed of 4/5 in the Home Frame (also called DS9 frame). The freighter passes DS9 (event A) at time $t = 0$ in both the freighter frame and the DS9 frame. At $t = 3$ h (according to DS9's clocks), Quark (a resident of DS9) sends a secret message to the freighter (event B). At time $t = 5$ h (according to DS9's clocks), the warp core of the Ferengi ship is mysteriously disabled (call this event C).

a) (2 points) Draw the spacetime diagram in the DS9 frame, indicating the worldlines of DS9 and the freighter, and events A, B, C. **Draw and label** the t' and x' axes for the freighter frame in this diagram. **Calibrate** these axes as carefully and accurately as you can.

We work in SR units(t and x measured in hours). The spacetime diagram with the worldlines for DS9, the freighter, events A,B,C is drawn in the Figure.



$$\beta = 4/5 \qquad \gamma = 5/3 \qquad \gamma\beta = 4/3 \tag{1}$$

Calibration:

$$\begin{aligned} \Delta x' &= \frac{1}{\gamma}\Delta x\,. \quad \text{Thus, for } \Delta x' = 1h\,, \quad \Delta x = \frac{5}{3}h \\ \Delta t' &= \frac{1}{\gamma}\Delta t\,. \quad \text{Thus, for } \Delta t' = 1h\,, \quad \Delta t = \frac{5}{3}h \end{aligned} \tag{2}$$

The two-observer diagram in the figure shows the DS9 frame, the worldlines of DS9 and the freighter, events A,B and C, and the calibrated axes of the freighter frame t' and x'. Note: each axis is labeled and units are also indicated.

b) (0.5 points) When and where does event B occur in the freighter frame? **Read these values from your diagram**, using dotted lines to project the event on the appropriate axes. Express the results in h (hours)

From the graph, one can to a good approximation conclude that event B occurs at $\Delta x'_{AB} \simeq -4h \quad \Delta t'_{AB} \simeq 5h$. Taking event A as the origin of the DS9 frame and the freighter frame, one has $t'_B \simeq 5h$ and $x'_B \simeq -4h$.

c) (1 points) Check the value of t'_B that you just found using the appropriate Lorentz transformation equation. **Show your work**

The Lorentz transformation equations from the DS9 frame (Home frame) to the freighter frame (Other frame) provide the solution. One obtains

$$\begin{aligned} \Delta x'_{AB} &= \gamma \Delta x_{AB} - \gamma\beta\Delta t_{AB} = -\gamma\beta\Delta t_{AB} = -4h \\ \Delta t'_{AB} &= \gamma \Delta t_{AB} - \gamma\beta\Delta x_{AB} = \gamma\Delta t_{AB} = 5h \end{aligned} \tag{3}$$

They tell that the coordinates of event B in the freighter frame (with origin at event A) are, exactly, $t'_B = 5h$ and $x'_B = -4h$. Hence, the approximate answer of the graphical method is in very good agrement with the exact answer given by the Lorentz transformations. More explanation, if correct, is always welcome.

d) (0.5 points) Which event, B or C, occurs first in the freighter frame?

Event C happens at $\Delta t_{AC} = 5h$ in the DS9 frame, and $\Delta x_{AC} = \beta\Delta t_{AC} = 4h$. Thus it happens at

$$\begin{aligned} \Delta x'_{AC} &= \gamma \Delta x_{AC} - \gamma\beta\Delta t_{AC} = 0 \\ \Delta t'_{AC} &= \gamma \Delta t_{AC} - \gamma\beta\Delta x_{AC} = 3h \end{aligned} \tag{4}$$

in the freighter frame. To summarize, $\Delta t'_{AC} = 3h$ and $\Delta t'_{AB} = 5h$. Hence, event C happens first in the freighter frame.

e) (1 points) The DS9 constable accuses Quark of *causing* the damage to the freighter by sending the signal. Is this possible? Explain why or why not.

The signal from B cannot travel faster than the speed of light. Thus, a signal sent from B will reach the freighter at time $t' > 5h$, later than when event C occurs ($t'_C = 3h$). In other words, events B and C are not causally connected: a signal from B to C would have to travel with $v > 1$ ($c = 1$ in SR units). Quark has not caused event C. One may add to the explanation above when exactly a signal sent at $v = 1$ at event B will reach the freighter.

More explanations over causality, light cone are welcome. For example, computing explicitly the spacetime interval: the spacetime interval between events B and C is spacelike, i.e. $\Delta s^2_{BC} < 0$, which means that events B and C are not causally connected and event B cannot have caused event C. Since the spacetime interval is frame independent, this conclusion is valid in any reference frame.

Another way of phrasing the same concept is by saying that event C is not inside the future light cone of event B, and thus they are not causally connected. The future light cone of event B contains all events that are connected to B by a signal traveling at velocity less than or equal the speed of light, and sent at event B (the past light cone contains events connected to B by a signal traveling at velocity less than or equal the speed of light, and received at event B).

2. (4 points) In a particle physics experiment, an unknown particle with an unknown mass M and initial velocity $\vec{u}$ decays into two identical particles of mass m. We observe that one of these final particles ends up essentially at rest, while the other moves in the $+x$ direction with speed $v = 12/13$. Use conservation of four-momentum to find the mass M and velocity $\vec{u}$ of the unknown particle, expressing the mass as a multiple of the known mass m. **Show your work**

4 points are distributed as follows:

- 2 points for writing the appropriate conservation law of the four-momentum in terms of its four components
- 2 points distributed over the sketch of the situation "before" and "after", and the correct algebra to obtain M and $\vec{u}$

Sketches go here.
Equations for the conservation of the four-momentum can also be written in the vector column

form (as done in the book). The equations are:

$$
\begin{aligned}
\frac{M}{\sqrt{1-u^2}} &= m + \frac{m}{\sqrt{1-v^2}} \\
\frac{M u_x}{\sqrt{1-u^2}} &= \frac{mv}{\sqrt{1-v^2}} \\
u_y &= 0 \\
u_z &= 0
\end{aligned}
\tag{5}
$$

Divide the second by the first to obtain

$$
u_x = \frac{v}{1+\sqrt{1-v^2}} = \frac{2}{3} \tag{6}
$$

From the second, one obtains

$$
M = m\frac{v}{u}\sqrt{\frac{1-u^2}{1-v^2}} = \frac{6}{\sqrt{5}}m \tag{7}
$$

where $u = |\vec{u}| = u_x$.

Note: do not use the vector symbol when you are instead indicating the components of the fourvector, $P_{t,x,y,z}$. Do not write $\vec{u} = 2/3$! A trivector cannot be equal to a number, because it contains three of them! You can say $|\vec{u}| = 2/3$. And, in this case, you should say $u_x = 2/3$, which also provides the information on the direction of motion of the initial particle with respect to the final particle with velocity v_x.

Tip: Avoid as much as possible the use of explicit numbers with decimal digits. Keep the formulas as clean as possible up to the end. This means that you first find an elegant formula that relates the unknown parameters to the known ones. Once you have done that, you find out the numerical result. At this point you are entitled to leave, e.g., $2/3$ or $6/\sqrt{5}$ in this case, indicated as your final answer (because this is the correct answer!). If you want to also write down the approximate value of $2/3$ with a given number of decimal digits, you can do it. But it is not needed. **Important:** Note that if you replace $2/3$ or $12/13$ or any fraction with their approximate vaue (up to a given decimal digit) inside the intermediate formulas, you will never obtain the exact answer! Instead, you are propagating the uncertainty caused by truncating to a given decimal digit through the calculation.